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INVARIANCE PRESSURE OF CONTROL SETS*

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Abstract. The invariance pressure of continuous-time control systems with initial states in a set K which are to be kept in a set Q is introduced and a number of results are derived, mainly for the case where Q is a control set.

Key words. invariance pressure, invariance entropy, control systems

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1. Introduction.

This paper generalizes the notion of invariance pressure introduced by the present authors in [1] and improves its characterization for linear control systems. We now admit initial states in a subset K of the set Q in which the system should remain. Then we derive a number of results for special sets Q, in particular, for control sets.

Invariance pressure is a generalization of invariance entropy for control systems by introducing a potential, similarly as topological pressure generalizes topological entropy for dynamical systems; cf., e.g., Walters [13]. Basic references for invariance entropy include Nair et al. [11], Colonius and Kawan [3], and the monograph Kawan [8]; cf. also Huang and Zhong [6] for relations to dimension theory and Da Silva [4] for the case of linear systems on Lie groups. This concept, similarly to other entropy concepts for control systems, like estimation entropy (Liberzon and Mitra [9]) and restoration entropy (Matveev and Pogromsky [12]) are introduced in order to analyze the minimal data rates in control tasks when data rate constraints are present (cf. also the monograph Matveev and Savkin [10]).

As main results we characterize the invariance pressure for the control set of linear control systems and for inner control sets we can show that the limit superior in the definition of invariance pressure can be replaced by the limit inferior.

The contents of this paper are as follows: After preliminaries on control systems and definitions of invariance pressure for admissible pairs (K, Q) in the sense of Kawan [8] in section 2, section 3 shows several basic dynamical properties of invariance pressure, in particular, its behavior under conjugacies. Section 4 shows that the outer invariance pressure and the invariance pressure coincide if the set Q is isolated, and the outer invariance pressure is discussed for sets Q which satisfy the no-return property. In particular, the invariance pressure is independent of the choice of a compact subset K with nonvoid interior in a control set D. Section 5 discusses the outer invariance pressure for inner control sets. Finally, section 6 derives an estimate for the invariance pressure of the control set of a linear control system.

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2. Preliminaries. In this section, we establish some notation and basic concepts for control systems which will be used throughout the paper.

2.1. Control systems. A continuous-time control system on a smooth manifold M is defined as a system

$$\Sigma \ : \ \dot{x}(t) = F(x(t), \omega(t)), \ \omega \in \mathcal{U},$$

where $\mathcal{U} := \{\omega : \mathbb{R} \to U;\ \omega$ is measurable with $\omega(t) \in U$ almost everywhere$\}$ is a set of *admissible control functions*, the *control range* U is a compact subset of $\mathbb{R}^m$, and $F : M \times \mathbb{R}^m \to TM$ is a C^1-map such that for each $u \in U$, $F_u(\cdot) := F(\cdot, u)$ is a vector field on M. For each $x \in M$ and $\omega \in \mathcal{U}$, we suppose that there exists a unique solution $\varphi(t, x, \omega)$ which is defined for all $t \in \mathbb{R}_+ = [0, \infty)$. We usually refer to the solution $\varphi(\cdot, x, \omega)$ as a *trajectory* of x with control function ω. We also fix a metric ρ on M which is compatible with the topology.

An important case in this paper is the linear control system (A, B) on $\mathbb{R}^d$, which is defined by

$$\Sigma_{lin} \ : \dot{x}(t) = Ax(t) + B\omega(t), \ \omega \in \mathcal{U},$$

where $A \in \mathbb{R}^{d\times d}$ and $B \in \mathbb{R}^{d\times m}$. We recall that the solution of this system is given by

$$\varphi(t, x, \omega) = e^{tA}x + \int_0^t e^{(t-s)A} B\omega(s) ds.$$

We need several notions characterizing controllability properties of subsets of the state space M of system Σ.

The *positive* and *negative orbits from* $x \in M$ are

$$\mathcal{O}^+(x) := \{y \in M; \ \ \text{there are } t > 0 \text{ and } \omega \in \mathcal{U} \text{ with } \varphi(t, x, \omega) = y\}$$

and

$$\mathcal{O}^-(x) := \{y \in M; \ \ \text{there are } t > 0 \text{ and } \omega \in \mathcal{U} \text{ with } \varphi(t, y, \omega) = x\},$$

respectively.

A set $Q \subset M$ is called *controlled invariant* if for all $x \in D$ there exists $\omega \in \mathcal{U}$ such that $\varphi(t, x, \omega) \in Q$ for all $t \geq 0$. We say that a set $Q \subset M$ satisfies the *no-return property* if

$$\forall x \in Q \ \forall \tau > 0 \ \forall \omega \in \mathcal{U} : \varphi(\tau, x, \omega) \in Q \Rightarrow \varphi([0, \tau], x, \omega) \subset Q.$$

A controlled invariant set $D \subset M$ is called a *control set* if it satisfies $D \subset \overline{\mathcal{O}^+(x)}$ for all $x \in D$ (approximate controllability) and D is a maximal controlled invariant set with this property. Note that for a control set with nonvoid interior the control set as well as its interior satisfy the no-return property.

2.2. Invariance pressure. Now we introduce the main concepts of the paper, the invariance and outer invariance pressure generalizing the concepts introduced in Colonius, Cossich and Santana [1].

A pair (K, Q) of nonempty subsets of M is called an *admissible pair* if K is compact and for each $x \in K$ there exists $\omega \in \mathcal{U}$ such that $\varphi(\mathbb{R}_+, x, \omega) \subset Q$ (in particular, $K \subset Q$).

Given an admissible pair (K, Q) and $\tau > 0$, we say that $\mathcal{S} \subset \mathcal{U}$ is a (τ, K, Q)-*spanning set* if for all $x \in K$ there is $\omega \in \mathcal{S}$ with $\varphi(t, x, \omega) \in Q$ for all $t \in [0, \tau]$. Let $C(U, \mathbb{R})$ denote the set of continuous functions $f : U \to \mathbb{R}$.

For $f \in C(U, \mathbb{R})$ denote $(S_\tau f)(\omega) := \int_0^\tau f(\omega(t))dt$ and

$$a_\tau(f, K, Q) := \inf \left\{ \sum_{\omega \in \mathcal{S}} e^{(S_\tau f)(\omega)};\ \mathcal{S}\ (\tau, K, Q)\text{-spanning} \right\}.$$

The *invariance pressure* $P_{inv}(f, K, Q; \Sigma)$ of control system Σ is given by

$$P_{inv}(f, K, Q; \Sigma) = \limsup_{\tau \to \infty} \frac{1}{\tau} \log a_\tau(f, K, Q).$$

To simplify the notation we use $P_{inv}(f, K, Q)$ when the considered control system is clear and, if $K = Q$, we omit the argument K and write $a_\tau(f, Q)$ and $P_{inv}(f, Q)$. Note that, in this case, we assume that Q is compact and controlled invariant.

Given an admissible pair (K, Q) such that Q is closed in M, and a metric ϱ on M, we define the *outer invariance pressure* of (K, Q) by

$$P_{out}(f, K, Q) = P_{out}(f, K, Q; \varrho; \Sigma) := \lim_{\varepsilon \to 0} P_{inv}(f, K, N_\varepsilon(Q)),$$

where $N_\varepsilon(Q) = \{y \in M;\ \exists\, x \in Q \text{ with } \varrho(x, y) < \varepsilon\}$ denotes the ε-neighborhood of Q.

Note that the limit for $\varepsilon \to 0$ exists and equals the supremum over $\varepsilon > 0$, since from Proposition 3 it follows that the pairs $(K, N_\varepsilon(Q))$ are admissible and that $\varepsilon_1 < \varepsilon_2$ implies $P_{inv}(f, K, N_{\varepsilon_1}(Q)) \geq P_{inv}(f, K, N_{\varepsilon_2}(Q))$. Furthermore, $P_{out}(f, K, Q) \leq P_{inv}(f, K, Q) \leq \infty$ for every admissible pair (K, Q) and $f \in C(U, \mathbb{R})$.

3. Properties of invariance pressure. In the first part of this section we study several properties of the invariance pressure and outer invariance pressure. In the second part of this section we show that conjugations preserve the invariance pressure.

3.1. Dynamical properties. We say that two metrics ϱ_1 and ϱ_2 on M are uniformly equivalent on Q if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in Q$ and for all $y \in M$ with $\varrho_i(x, y) < \delta$ implies that $\varrho_j(x, y) < \varepsilon$ for $i, j = 1, 2$, $i \neq j$.

The following proposition states that the value of the outer invariance pressure of (K, Q) does not change when we consider uniformly equivalent metrics. Since the proof is similar to Kawan [8, Proposition 2.1.12], we will omit it.

PROPOSITION 1. *Let (K, Q) be an admissible pair such that Q is closed in M. If ϱ_1 and ϱ_2 are two metrics on M which are uniformly equivalent on Q, then $P_{out}(f, K, Q; \varrho_1) = P_{out}(f, K, Q; \varrho_2)$ for all $f \in C(U, \mathbb{R})$. If Q is compact, then this is automatically satisfied, and in this case the outer invariance pressure is independent of the metric.*

The next proposition shows that we just need finite spanning sets to get $a_\tau(f, K, Q)$ and it is a reformulation of [1, Proposition 5].

PROPOSITION 2. *Consider an admissible pair (K, Q) with Q open in M and $f \in C(U, \mathbb{R})$. Then*

$$a_\tau(f, K, Q) = \inf \left\{ \sum_{\omega \in \mathcal{S}} e^{(S_n f)(\omega)};\ \mathcal{S} \textit{ is a finite } (\tau, K, Q)\textit{-spanning set} \right\}.$$

Proof. Since Q is open, $\varphi(t,\cdot,\omega)$ is continuous for all $t\in\mathbb{R}$ and $\omega\in\mathcal{U}$, and K is compact, every (τ,K,Q)-spanning set $\mathcal{S}$ admits a finite (τ,K,Q)-spanning subset $\mathcal{S}'\subset\mathcal{S}$. Now define

$$\widetilde{a}_\tau(f,K,Q):=\inf\left\{\sum_{\omega\in\mathcal{S}}e^{(S_nf)(\omega)};\ \mathcal{S}\text{ is a finite }(\tau,K,Q)\text{-spanning set}\right\}.$$

Since clearly $a_\tau(f,K,Q)\le\widetilde{a}_\tau(f,K,Q)$, we just have to prove the reverse inequality. Given a (τ,K,Q)-spanning set $\mathcal{S}$, as shown earlier, there is a finite (τ,K,Q)-spanning subset $\mathcal{S}'\subset\mathcal{S}$. Hence $\sum_{\omega\in\mathcal{S}'}e^{(S_\tau f)(\omega)}\le\sum_{\omega\in\mathcal{S}}e^{(S_\tau f)(\omega)}$, which implies that $\widetilde{a}_\tau(f,K,Q)\le a_\tau(f,K,Q)$. ◻

The next results of this section show several basic properties that help to understand the concept of invariance pressure.

Proposition 3. *The following assertions hold for an admissible pair (K,Q):*

(i) *If $0<\tau_1<\tau_2$ and $f\ge 0$, then $a_{\tau_1}(f,K,Q)\le a_{\tau_2}(f,K,Q)$.*
(ii) *If $Q\subset R$, then (K,R) is admissible and $a_\tau(f,K,Q)\ge a_\tau(f,K,R)$; hence $P_{inv}(f,K,Q)\ge P_{inv}(f,K,R)$.*
(iii) *If $L\subset K$ is closed in M, then (L,Q) is admissible and $a_\tau(f,L,Q)\le a_\tau(f,K,Q)$; hence, $P_{inv}(f,L,Q)\le P_{inv}(f,K,Q)$.*
(iv) *Let $\Sigma':\dot y(t)=F'(y(t),\omega(t))$ be another system in M with solution φ' and a set of admissible control functions $\mathcal{U}'$ containing $\mathcal{U}$ and $\varphi'(t,x,\omega)=\varphi(t,x,\omega)$ whenever $\omega\in\mathcal{U}$. Then (K,Q) is also admissible for Σ' and $P_{inv}(f,K,Q;\Sigma')\le P_{inv}(f,K,Q;\Sigma)$.*

Proposition 4. *The following assertions hold for an admissible pair (K,Q), functions $f,g\in C(U,\mathbb{R})$, and $c\in\mathbb{R}$:*

(i) *$P_{inv}(\mathbf{0},K,Q)=h_{inv}(K,Q)$, where $\mathbf{0}$ is the null function in $C(U,\mathbb{R})$.*
(ii) *If $f\le g$, then $P_{inv}(f,K,Q)\le P_{inv}(g,K,Q)$. In particular $h_{inv}(K,Q)+\inf f\le P_{inv}(f,K,Q)\le h_{inv}(K,Q)+\sup f$.*
(iii) *$P_{inv}(f+c,K,Q)=P_{inv}(f,K,Q)+c$.*
(iv) *$|P_{inv}(f,K,Q)-P_{inv}(g,K,Q)|\le\|f-g\|_\infty$.*

Proof. (i) and (ii) are clear from the definition of invariance pressure.
The statements (iii) and (iv) follow analogously to Proposition 13(ii) of [1]. ◻

The next corollaries deal with the finiteness of invariance pressure.

Corollary 5. *Consider $f\in C(U,\mathbb{R})$. Then*

(i) *if Q is open, $a_\tau(f,K,Q)$ is finite for all $\tau>0$;*
(ii) *if Q is a compact controlled invariant set, $a_\tau(f,Q)$ is either finite for all $\tau>0$ or for none.*

Proof. The two statements follow from the inequalities

$$e^{\tau\inf f}r_{inv}(\tau,K,Q)\le a_\tau(f,K,Q)\le e^{\tau\sup f}r_{inv}(\tau,K,Q)$$

and [8, Propositions 2.2 and 2.3(i)]. ◻

Remark 6. Note that Propositions 2 and 4 and Corollary 5(i) also hold for outer invariance pressure.

As an immediate consequence, we have the following.

Corollary 7. *If $f \in C(U,\mathbb{R})$ and Q is compact, then the following assertions are equivalent:*

(i) *$P_{inv}(f,Q)$ is finite;*
(ii) *$a_\tau(f,Q)$ is finite for some τ;*
(iii) *$a_\tau(f,Q)$ is finite for all τ.*

Proposition 8. *If Q is a compact controlled invariant set and $f \in C(U,\mathbb{R})$, then the function $\tau \mapsto a_\tau(f,Q)$ is subadditive and therefore*

$$P_{inv}(f,Q) = \lim_{\tau\to\infty} \frac{1}{\tau}\log a_\tau(f,Q) = \inf_{\tau>0} \frac{1}{\tau}\log a_\tau(f,Q).$$

Proof. If $a_\tau(f,Q) = \infty$ for all τ, the assertion is trivial. Hence, by Corollary 5(ii) we can assume that $a_\tau(f,Q) < \infty$ for all τ. If we show that $a_{\tau_1+\tau_2}(f,Q) \leq a_{\tau_1}(f,Q) \cdot a_{\tau_2}(f,Q)$ for all $\tau_1,\tau_2 > 0$, then the result follows from the subadditivity lemma; see, e.g., [8, Lemma B.7.1]. To this end, consider for $j = 1,2$, (τ_j,Q)-spanning sets $\mathcal{S}_j$. For $\omega_1 \in \mathcal{S}_1, \omega_2 \in \mathcal{S}_2$ define a control function $\omega \in \mathcal{U}$ by

$$\omega(t) = \begin{cases} \omega_1(t) & \text{if} \quad t \in [0,\tau_1], \\ \omega_2(t-\tau_1) & \text{if} \quad t > \tau_1. \end{cases}$$

These functions form a $(\tau_1+\tau_2,Q)$-spanning set. Hence $a_{\tau_1+\tau_2}(f,Q) \leq a_{\tau_1}(f,Q) \cdot a_{\tau_2}(f,Q)$, which concludes the proof. ∎

3.2. Invariance pressure under conjugacy. Now we show that for systems that are conjugate the respective invariance pressures coincide.

Definition 9. *Consider two control systems*

$$\Sigma_1 : \ \dot{x}(t) = F_1(x(t),\omega(t)) \text{ and } \Sigma_2 : \ \dot{y}(t) = F_2(y(t),\nu(t))$$

on M_1 and M_2 with compact control ranges U_1 and U_2, sets of control functions $\mathcal{U}_1$ and $\mathcal{U}_2$, and solutions φ_1 and φ_2, respectively. Let $\pi : \mathbb{R}_+ \times M_1 \to M_2$, $(t,x) \mapsto \pi_t(x)$, and $H : U_1 \to U_2$ be continuous maps such that the induced map $h_H : \mathcal{U}_1 \to \mathcal{U}_2$, $h_H(\omega)(t) := H(\omega(t))$ for all $t \in \mathbb{R}$, satisfies

$$\pi_t(\varphi_1(t,x,\omega)) = \varphi_2(t,\pi_0(x),h_H(\omega)) \text{ for all } t \in \mathbb{R}_+, x \in M_1, \text{ and } \omega \in \mathcal{U}_1.$$

Then (π,H) is called a time-variant semiconjugacy *from Σ_1 to Σ_2. If each of the maps $\pi_t : M_1 \to M_2$ and $H : U_1 \to U_2$ are homeomorphisms, we call (π,H) a* time-variant conjugacy *from Σ_1 to Σ_2.*

Analogously we define a time-invariant semiconjugacy *and* conjugacy *from Σ_1 to Σ_2 if π is independent of $t \in \mathbb{R}_+$.*

Proposition 10. *Consider two systems as in Definition 9 and let (π,H) be a time-variant semiconjugacy from Σ_1 to Σ_2. Further assume that (K,Q) is an admissible pair for Σ_1 and*

$$\pi_t(Q) \subset \pi_0(Q) \quad \text{for all } t > 0.$$

Then $(\pi_0(K),\pi_0(Q))$ is an admissible pair for system Σ_2 and

$$P_{inv}(f \circ H,K,Q;\Sigma_1) \geq P_{inv}(f,\pi_0(K),\pi_0(Q));\Sigma_2)$$

for all $f \in C(U_2,\mathbb{R})$. Moreover, if Q is compact and the family $\{\pi_t\}_{t\in\mathbb{R}_+}$ is pointwise *equicontinuous, then*

$$P_{out}(f \circ H, K, Q; \Sigma_1) \geq P_{out}(f, \pi_0(K), \pi_0(Q)); \Sigma_2)$$

for all $f \in C(U_2, \mathbb{R})$.

Proof. In order to show that $(\pi_0(K), \pi_0(Q))$ is an admissible pair, note that since π is continuous, the set $\pi_0(K)$ is compact. Let $y \in \pi_0(K)$, then $y = \pi_0(x)$ for some $x \in K$. Since (K, Q) is an admissible pair, there is $\omega \in \mathcal{U}_1$ such that $\varphi(\mathbb{R}_+, x, \omega) \subset Q$, and we obtain

$$\varphi_2(t, y, h_H(\omega)) = \pi_t(\varphi_1(t, x, \omega)) \in \pi_t(Q) \subset \pi_0(Q).$$

Therefore $(\pi_0(K), \pi_0(Q))$ is an admissible pair for Σ_2.

Now, let $\mathcal{S} \subset \mathcal{U}_1$ be a (τ, K, Q)-spanning set. With the same arguments as above, we find that $h_H(\mathcal{S}) \subset \mathcal{U}_2$ is $(\tau, \pi_0(K), \pi_0(Q))$-spanning. Hence

$$\sum_{\mu \in h_H(\mathcal{S})} e^{(S_\tau f)(\mu)} = \sum_{\omega \in \mathcal{S}} e^{(S_\tau f)(H \circ \omega)} = \sum_{\omega \in \mathcal{S}} e^{(S_\tau (f \circ H))(\omega)}$$

for every (τ, K, Q)-spanning set $\mathcal{S}$, which implies that

$$a_\tau(f, \pi_0(K), \pi_0(Q)) \leq a_\tau(f \circ H, K, Q).$$

Therefore $P_{inv}(f, \pi_0(K), \pi_0(Q)) \leq P_{inv}(f \circ H, K, Q)$.

Now assume that Q is compact. Let ϱ_1 denote a metric on M_1 and ϱ_2 a metric on M_2. By compactness of Q, the pointwise equicontinuity of $\{\pi_t\}_{t \in \mathbb{R}_+}$ on Q is uniform, hence, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $t \in \mathbb{R}_+$, $x \in Q$, and $y \in M_1$ the condition $\varrho_1(x, y) < \delta$ implies $\varrho_2(\pi_t(x), \pi_t(y)) < \varepsilon$.

Let $\mathcal{S} \subset \mathcal{U}_1$ be a $(\tau, K, N_\delta(Q))$-spanning set with $\delta = \delta(\varepsilon)$ as above. Note that if $y \in \pi_0(K)$, then $y = \pi_0(x)$ for some $x \in K$. For $\omega \in \mathcal{S}$ such that $\varphi_1([0, \tau], x, \omega) \subset N_\delta(Q)$ and for each $t \in [0, \tau]$, there exists $x_t \in Q$ with $\varrho_1(x_t, \varphi_1(t, x, \omega)) < \delta$. This implies that for all $t \in [0, \tau]$

$$\varrho_2(\varphi_2(t, y, h_H(\omega)), \pi_t(x_t)) = \varrho_2(\pi_t(\varphi_1(t, x, \omega)), \pi_t(x_t)) < \varepsilon.$$

This shows that $h_H(\mathcal{S}) \subset \mathcal{U}_2$ is a $(\tau, \pi_0(K), N_\varepsilon(\pi_0(Q)))$-spanning set. We conclude that $a_\tau(f, \pi_0(K), N_\varepsilon(\pi_0(Q))) \leq a_\tau(f \circ H, K, N_\varepsilon(Q))$ and, hence,

$$P_{out}(f, \pi_0(K), \pi_0(Q)) \leq P_{out}(f \circ H, K, Q). \qquad \square$$

Remark 11. It is easy to see that if (π, H) is a time-variant conjugacy from Σ_1 to Σ_2, then (ψ, H^{-1}) with $\psi_t(y) := \pi_t^{-1}(y)$ is a time-variant conjugacy from Σ_2 to Σ_1. In this case, we have, under the assumptions of the previous proposition,

$$P_{inv}(f \circ H, K, Q; \Sigma_1) = P_{inv}(f, \pi_0(K), \pi_0(Q)); \Sigma_2).$$

A similar argument holds for time-invariant conjugacies.

Example 12. Consider two linear control systems $\mathbb{R}^d$,

$$\Sigma_1 : \ \dot{x}(t) = A_1 x(t) + B_1\omega(t) \text{ and } \ \Sigma_2 : \ \dot{x}(t) = A_2 x(t) + B_2\omega(t),$$

where $\omega(t)$ is in a compact set $U \subset \mathbb{R}^m$ for all $t \in \mathbb{R}$, $A_i \in \mathbb{R}^{d\times d}$, and $B_i \in \mathbb{R}^{d\times m}$ for $i = 1, 2$. If there is a nonsingular $d \times d$ matrix T such that $A_2 = TA_1T^{-1}$ and $B_2 = TB_1$, then (T, id_U) is a time-invariant conjugacy from Σ_1 to Σ_2. In fact,

$$\begin{aligned} T\left(\varphi_1(t,x,\omega)\right) &= T \quad e^{tA_1}x + \int_0^t e^{(t-s)A_1}B_1\omega(s)ds \\ &= T \quad e^{tT^{-1}A_2T}x + \int_0^t e^{(t-s)T^{-1}A_2T}T^{-1}B_2\omega(s)ds \\ &= T \quad T^{-1}e^{tA_2}Tx + \int_0^t T^{-1}e^{(t-s)A_2}TT^{-1}B_2\omega(s)ds \\ &= e^{tA_2}Tx + \int_0^t e^{(t-s)A_2}B_2\omega(s)ds = \varphi_2(t, Tx, h_{id_U}(\omega)). \end{aligned}$$

In this case, it follows for every admissible pair (K, Q) and all $f \in C(U, \mathbb{R})$

$$P_{inv}(f, K, Q; \Sigma_1) = P_{inv}(f, T(K), T(Q); \Sigma_2).$$

4. Outer invariance pressure on special sets. In this section, we study the outer invariance pressure in isolated sets and in sets satisfying the no-return property. In the first case, we will see that the limit for $\varepsilon \to 0$ in the definition of $P_{out}(K, Q)$ becomes superfluous and, in the second case, we will obtain that the limit superior in this definition can be replaced by a limit inferior.

We assume that the system Σ satisfies the following additional properties:

(1) The set $\mathcal{U}$ of admissible control functions is endowed with a topology that makes it a sequentially compact space, that is, every sequence in $\mathcal{U}$ has a convergent subsequence.

(2) The solution map $\varphi : \mathbb{R}_+ \times M \times \mathcal{U} \to M$ is continuous when $\mathcal{U}$ is endowed with the above topology.

These properties are satisfied, in particular, for a control-affine system of the form

$$\dot{x}(t) = X_0(x) + \sum_{i=1}^{m} u_i X_i(x),$$

where $X_0, X_1, \ldots, X_m$ are C^1 vector fields and $u = (u_1, \ldots, u_m) \in U \subset \mathbb{R}^m$ with U compact and convex. Then the appropriate topology on $\mathcal{U}$ is the weak*-topology of $L^\infty(\mathbb{R}; \mathbb{R}^m) = L^1(\mathbb{R}; \mathbb{R}^m)^*$.

A compact set $Q \subset M$ is called *isolated* if there exists $\delta_0 > 0$ such that for all $(x, \omega) \in \overline{N_{\delta_0}(Q)} \times \mathcal{U}$ the following implication holds:

$$\varphi(\mathbb{R}_+, x, \omega) \subset \overline{N_{\delta_0}(Q)} \Rightarrow \varphi(\mathbb{R}_+, x, \omega) \subset Q. \tag{4.1}$$

Proposition 13. *Let (K, Q) be an admissible pair such that Q is compact and isolated with constant δ_0. Then it holds, for all $f \in C(U, \mathbb{R})$,*

$$P_{out}(f, K, Q) = P_{inv}(f, K, N_\varepsilon(Q)) \textit{ for all } \varepsilon \in (0, \delta_0].$$

Proof. Since M is locally compact, by Kawan [8, Lemma A.4.2] we may assume that δ_0 is small enough that $\overline{N_{\delta_0}(Q)}$ is compact, since assumption (4.1) is also satisfied for smaller δ_0.

By an argument similar to [8, Proposition 2.2.17], we can see that for all $\rho > 0$ and for all $\varepsilon \in (0, \delta_0]$ there is $n \in \mathbb{N}$ such that for all $(x, \omega) \in \overline{N_{\delta_0}(Q)} \times \mathcal{U}$ $\max_{t\in[0,n]} \operatorname{dist}(\varphi(t, x, \omega), Q) \leq \varepsilon$ implies $\operatorname{dist}(x, Q) < \rho$.

Now let $0 < \varepsilon_1 < \varepsilon_2 \leq \delta_0$. Then there exists $n \in \mathbb{N}$ such that for all $(x, \omega) \in \overline{N_{\delta_0}(Q)} \times \mathcal{U}$ it holds that $\max_{t\in[0,n]} \operatorname{dist}(\varphi(t, x, \omega), Q) \leq \varepsilon_2$ implies $\operatorname{dist}(x, Q) < \varepsilon_1$. For arbitrary $\tau > 0$, let $\mathcal{S}$ be an $(n + \tau, K, N_{\varepsilon_2}(Q))$-spanning set. For $x \in K$, there exists $\omega_x \in \mathcal{S}$ with $\varphi([0, n + \tau], x, \omega_x) \subset N_{\varepsilon_2}(Q)$. For every $s \in [0, \tau]$, we obtain $\max_{t\in[0,n]} \operatorname{dist}(\varphi(t, \varphi(s, x, \omega_x), \Theta_s\omega_x), Q) = \max_{t\in[0,n]} \operatorname{dist}(\varphi(t + s, x, \omega_x), Q) < \varepsilon_2$. Hence we have $\operatorname{dist}(\varphi(s, x, \omega_x), Q) < \varepsilon_1$ for all $s \in [0, \tau]$, which implies that $\mathcal{S}$ is a $(\tau, K, N_{\varepsilon_1}(Q))$-spanning set. Therefore, given $g \in C(U, \mathbb{R})$, $g \geq 0$, we get

$$a_\tau(g, K, N_{\varepsilon_1}(Q)) \leq a_{n+\tau}(g, K, N_{\varepsilon_2}(Q)) \ \forall \tau > 0,$$

which implies $P_{inv}(f, K, N_{\varepsilon_1}(Q)) \leq P_{inv}(f, K, N_{\varepsilon_2}(Q))$, for all $f \in C(U, \mathbb{R})$.

By Proposition 3(ii) we have $P_{inv}(f, K, N_{\varepsilon_2}(Q)) \leq P_{inv}(f, K, N_{\varepsilon_1}(Q))$ and the proof is complete. $\square$

PROPOSITION 14. *Let $Q \subset M$ be a set with the no-return property. Assume that (K_1, Q) and (K_2, Q) are two admissible pairs such that K_2 has a nonempty interior and*

$$\forall x \in K_1 \ \exists \omega_x \in \mathcal{U} \ \exists \tau_x > 0 : \varphi(\tau_x, x, \omega_x) \in \operatorname{int} K.$$

Then for all $f \in C(U, \mathbb{R})$

$$P_{inv}(f, K_1, Q) \leq P_{inv}(f, K_2, Q).$$

Proof. Note that if there exists τ_0 such that $a_\tau(f, K_2, Q) = +\infty$ for all $\tau \geq \tau_0$, then $P_{inv}(f, K_2, Q) = +\infty$ and hence the assertion holds.

If this is not the case, we can get a sequence $\tau_k \to \infty$ such that $a_{\tau_k}(f, K_2, Q)$ is finite for all k. For all $x \in K_1$, let $\omega_x \in \mathcal{U}$ and $\tau_x > 0$ as in the assumption. Since $\varphi(\tau_x, \cdot, \omega_x)$ is continuous, we find, for every $x \in K_1$, an open neighborhood V_x of x such that $\varphi(\tau_x, V_x, \omega_x) \subset \operatorname{int} K_2$. By the no-return property of Q, we have $\varphi([0, \tau_x], y, \omega_x) \subset Q$, for all $y \in K_1 \cap V_x$. The family $\{V_x\}_{x\in K_1}$ is an open cover of K_1 and by compactness there exist $x_1, \ldots, x_n \in K_1$ with $K_1 \subset \cup_{i=1}^n V_{x_i}$. Now, let $\mathcal{S} := \{\mu_1, \ldots, \mu_k\}$ be a finite (τ, K_2, Q)-spanning set for some $\tau > \tau_M - \tau_m$, where $\tau_M := \max_{1\leq i\leq n} \tau_{x_i}$ and $\tau_m := \min_{1\leq i\leq n} \tau_{x_i}$.

For every index pair (i, j) with $1 \leq i \leq n$, $1 \leq j \leq k$ such that there exists $x \in K_1$ with $y_x := \varphi(\tau_{x_i}, x, \omega_{x_i}) \in \operatorname{int} K_2$ and $\varphi([0, \tau], y_x, \mu_j) \subset Q$, we can define a control function

$$\nu_{ij}(t) = \begin{array}{rl} \omega_{x_i}(t) & \text{if} \quad t \in [0, \tau_{x_i}], \\ \mu_j(t - \tau_{x_i}) & \text{if} \quad t > \tau_{x_i}. \end{array}$$

Define the set $\widetilde{\mathcal{S}}$ of all these control functions. Let $\widetilde{\tau} := \tau + \tau_m$, hence $\tau \geq \widetilde{\tau} - \tau_M$. Then $\widetilde{\mathcal{S}}$ is a $(\widetilde{\tau}, K_1, Q)$-spanning set by construction and, consequently, for all $f \in C(U, \mathbb{R})$,

$f \geq 0$, we have

$$\begin{aligned}(S_{\widetilde{\tau}}f)(\nu_{ij}) &= (S_{\tau_{x_i}}f)(\omega_{x_i}) + \int_{\tau_{x_i}}^{\widetilde{\tau}} f(\mu_j(t-\tau_{x_i}))dt \\ &= (S_{\tau_{x_i}}f)(\omega_{x_i}) + \int_0^{\widetilde{\tau}-\tau_{x_i}} f(\mu_j(t))dt \\ &= (S_{\tau_{x_i}}f)(\omega_{x_i}) + (S_{\widetilde{\tau}-\tau_{x_i}}f)(\mu_j) \\ &\leq (S_{\tau_{x_i}}f)(\omega_{x_i}) + (S_\tau f)(\mu_j).\end{aligned}$$

Hence

$$\begin{aligned}a_\tau(f,K_1,Q) \leq a_{\widetilde{\tau}}(f,K_1,Q) \leq \sum_{\nu_{ij}\in\widetilde{\mathcal{S}}} e^{(S_{\widetilde{\tau}}f)(\nu_{ij})} &\leq \sum_{1\leq i\leq n,\ \mu\in\mathcal{S}} e^{(S_{\tau_{x_i}}f)(\omega_{x_i})}e^{(S_\tau f)(\mu)} \\ \leq \sum_{1\leq i\leq n} e^{(S_{\tau_{x_i}}f)(\omega_{x_i})} \cdot \sum_{\mu\in\mathcal{S}} e^{(S_\tau f))(\mu)} &\leq ne^{\|f\|\tau_M} \sum_{\mu\in\mathcal{S}} e^{(S_\tau f(\mu)},\end{aligned}$$

because $0 \leq \widetilde{\tau} - \tau_{x_i} \leq \tau$. Since this inequality holds for all finite (τ, K_2, Q)-spanning sets, we have

$$a_\tau(f,K_1,Q) \leq ne^{\|f\|\tau_M} a_\tau(f,K_2,Q),\ \tau > \tau_M - \tau_m.$$

Therefore, we obtain for all $f \in C(U,\mathbb{R})$, $f \geq 0$,

$$P_{inv}(f,K_1,Q) \leq P_{inv}(f,K_2,Q).$$

Now consider an arbitrary $f \in C(U,\mathbb{R})$. Then $\tilde{f} \in C(U,\mathbb{R})$ given by $\tilde{f}(u) = f(u) - \inf f$ satisfies $\tilde{f} \geq 0$. Using Proposition 4(iii) it follows that

$$\begin{aligned}P_{inv}(f,K_1,Q) &= P_{inv}(\tilde{f},K_1,Q) + \inf_{u\in U} f(u) \leq P_{inv}(\tilde{f},K_2,Q) + \inf_{u\in U} f(u) \\ &= P_{inv}(f,K_2,Q) - \inf_{u\in U} f(u) + \inf_{u\in U} f(u) \\ &= P_{inv}(f,K_2,Q).\end{aligned}$$

□

COROLLARY 15. *Let $D \subset M$ be a control set and let $K_1, K_2 \subset D$ be two compact subsets with nonempty interior. Then (K_1, D) and (K_2, D) are admissible pairs and for all $f \in C(U,\mathbb{R})$ we have*

$$P_{inv}(f,K_1,D) = P_{inv}(f,K_2,D).$$

Proof. This follows, since control sets with a nonvoid interior satisfy the no-return property. □

5. Outer invariance pressure for inner control sets. In this section, we will show that the limit superior in the definition of invariance pressure of a control set can be replaced by the limit inferior, if certain controllability properties near the control set are satisfied.

A control set $D \subset M$ is called an *inner control set* if there exists an increasing family of compact and convex sets $\{U_\rho\}_{\rho\in[0,1]}$ in $\mathbb{R}^m$ (i.e., $U_{\rho_1} \subset U_{\rho_2}$ for $\rho_1 < \rho_2$), such that for every $\rho \in [0,1]$ the system Σ with control range U_ρ (instead of U) has a control set D_ρ with nonvoid interior and compact closure, and the following conditions are satisfied:

(i) $U = U_0$ and $D = D_0$;

(ii) $\overline{D_{\rho_1}} \subset \operatorname{int}(D_{\rho_2})$ whenever $\rho_1 < \rho_2$;

(iii) for every neighborhood W of $\overline{D}$ there is $\rho \in [0,1)$ with $\overline{D_\rho} \subset W$.

This notion (slightly modified) is taken from Kawan [8, Definition 2.6]. Below, we will consider an inner control set $D = D_0$ (corresponding to the control range $U = U_0$) and characterize the invariance pressure of the controlled invariant set $Q = \overline{D}$ with respect to the larger control range $U_1 \supset U_0$.

The following result shows that for admissible pairs (K,Q), where Q is the closure of an inner control set, the limit superior in the definition of outer invariance pressure can be replaced by the limit inferior. The proof follows [8, Proposition 2.16].

PROPOSITION 16. *Let Q be the closure of an inner control set D of a system Σ. Then for every compact set $K \subset D$, the pair (K,Q) is admissible for the system with control range U_1 and, if* $\operatorname{int}K \neq \emptyset$, *we have*

$$P_{out}(f,K,Q) = \lim_{\varepsilon\searrow 0}\liminf_{\tau\to\infty}\frac{1}{\tau}\log a_\tau(f,K,N_\varepsilon(Q)) \text{ for every } f \in C(U,\mathbb{R}).$$

Proof. First observe that (by the Tietze extension theorem) every continuous function $f \in C(U,\mathbb{R})$ can be extended to a continuous function $f \in C(U_1,\mathbb{R})$. We fix such an extension. Our proof will show that $P_{out}(f,K,Q)$ does not depend on this extension.

From conditions (ii) and (iii) of inner control sets, it follows that there exists a monotonically decreasing sequence $(\rho_n)_{n\in\mathbb{N}}$ in $[0,1)$ with $D_{\rho_n} \subset N_{1/n}(Q)$ for all $n \in \mathbb{N}$. Since $Q = \overline{D} \subset \operatorname{int}(D_{\rho_n})$ for all $n \in \mathbb{N}$, we can find a monotonically decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive real numbers with $\varepsilon_n \searrow 0$ such that $\overline{N_{\varepsilon_n}(Q)} \subset D_{\rho_n}$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ it is possible to steer all points of $N_{\varepsilon_n}(Q)$ to K with finitely many control functions using the control range U_{ρ_n}. In fact, since $\overline{N_{\varepsilon_n}(Q)}$ and K are subsets of the control set D_{ρ_n} for each n, then for all $x \in \overline{N_{\varepsilon_n}(Q)}$, there exist $t^n_x > 0$ and $\mu^n_x \in \mathcal{U}$, $\mu^n_x(t) \in U_{\rho_n}$ for all t, such that $\varphi(t^n_x, x, \mu^n_x) \in \operatorname{int}K$ by the approximate controllability of the control set D_{ρ_n}. Continuity implies that there exists a neighborhood W^n_x of x such that $\varphi(t^n_x, W^n_x, \mu^n_x) \subset \operatorname{int}K$. By compactness there exist $x^n_1,\dots,x^n_{k_n} \in \overline{N_{\varepsilon_n}(Q)}$ such that

$$\overline{N_{\varepsilon_n}(Q)} \subset \bigcup_{i=1}^{k_n} W^n_{x_i}.$$

Denote $\mathcal{S}_n := \{\mu^n_1,\dots,\mu^n_{k_n}\}$, where $\mu^n_j = \mu^n_{x_j}$, and $\tau^n_j := t^n_{x_j}$. Observe that given $x \in N_{\varepsilon_n}(Q)$, the trajectory $\varphi(t,x,\mu^n_j)$, $t \in [0,\tau^n_j]$, does not leave the control set $D_{\rho_n} \subset N_{1/n}(Q)$ by the no-return property.

For every $\tau > \tau^n_M := \max\{\tau^n_j; j = 1,\dots,k_n\}$ consider a finite $(\tau,K,N_\varepsilon(Q))$-spanning set $\mathcal{S} = \{\omega_1,\dots,\omega_k\}$, where $\varepsilon \in (0,\varepsilon_n]$ and the controls take values in U_0. Let $\widetilde{\mathcal{S}}$ be the set consisting of the functions

$$\nu^n_{ij}(t) = \begin{array}{rll} \omega_i(t) & \text{if} & t \in [0,\tau-\tau^n_j], \\ \mu^n_j(t-\tau^n_j) & \text{if} & t \in (\tau-\tau^n_j,\tau], \end{array} \quad 1 \le i \le k \text{ and } 1 \le j \le k_n.$$

Thus for every $x \in K$ there is a control in $\tilde{S}$ keeping the corresponding trajectory in $N_\varepsilon(Q)$ up to time $\tau - \tau^n_j$ and then steering the system back to K. Now, for $m \in \mathbb{N}$, define $\widehat{\mathcal{S}}$ as the set obtained by m iterations of the elements of $\widetilde{\mathcal{S}}$. Hence $\widehat{\mathcal{S}}$ is an $(m\tau, K, N_{1/n}(Q))$-spanning set with $\#\widehat{\mathcal{S}} \leq (\#\mathcal{S})^m \cdot (\#\mathcal{S}_n)^m < \infty$.

We compute for $\nu \in \widehat{\mathcal{S}}$

$$\begin{aligned}
(S_{m\tau}f)(\nu) &= \int_0^{m\tau} f(\nu(t))dt = \int_0^{\tau} f(\nu_{i_1,j_1}(t))dt + \cdots + \int_{(m-1)\tau}^{m\tau} f(\nu_{i_m,j_m}(t))dt \\
&= \int_0^{\tau-\tau_{j_1}} f(\omega_{i_1}(t))dt + \int_{\tau-\tau_{j_1}}^{\tau} f(\mu^n_{j_1}(t-\tau^n_{j_1}))dt \\
&\quad + \cdots + \int_{(m-1)\tau}^{m\tau-\tau_{j_m}} f(\omega_{i_m}(t-(m-1)\tau))dt \\
&\quad + \int_{m\tau-\tau_{j_m}}^{m\tau} f(\mu^n_{j_m}(t-(m\tau-\tau^n_{j_m})))dt \\
&= \int_0^{\tau-\tau_{j_1}} f(\omega_{i_1}(t))dt + \int_0^{\tau_{j_1}} f(\mu^n_{j_1}(t))dt \\
&\quad + \cdots + \int_0^{\tau-\tau_{j_m}} f(\omega_{i_m}(t))dt + \int_0^{\tau_{j_m}} f(\mu^n_{j_m}(t))dt \\
&\leq (S_\tau f)(\omega_{i_1}) + \cdots + (S_\tau f)(\omega_{i_m}) + 2m\tau^n_M \sup f.
\end{aligned}$$

This implies for all $(\tau, K, N_\varepsilon(Q))$-spanning sets $\mathcal{S}$ and $\varepsilon \in (0, \varepsilon_n]$

$$\begin{aligned}
a_{m\tau}(f, K, N_{1/n}(Q)) &\leq \sum_{\nu \in \widehat{\mathcal{S}}} e^{(S_{m\tau}f)(\nu)} \\
&\leq e^{2m\tau^n_M \sup f} \cdot \sum_{\omega_{i_l} \in \mathcal{S};\ 1 \leq l \leq m} e^{(S_\tau f)(\omega_{i_1}) + \cdots + (S_\tau f)(\omega_{i_m})} \\
&\leq e^{2m\tau^n_M \sup f} \cdot \left(\sum_{\omega \in \mathcal{S}} e^{(S_\tau f)(\omega)} \right) \cdots \left(\sum_{\omega \in \mathcal{S}} e^{(S_\tau f)(\omega)} \right) \\
&= e^{2m\tau^n_M \sup f} \cdot \left(\sum_{\omega \in \mathcal{S}} e^{(S_\tau f)(\omega)} \right)^m .
\end{aligned}$$

It follows that $a_{m\tau}(f, K, N_{1/n}(Q)) \leq e^{2m\tau^n_M \sup f} \cdot (a_\tau(f, K, N_\varepsilon(Q)))^m$ for all $m \in \mathbb{N}$, $\tau > 0$, and $\varepsilon \in (0, \varepsilon_n]$. By discretization of time we get

$$\begin{aligned}
P_{inv}(f, K, N_{1/n}(Q)) &= \limsup_{m\to\infty} \frac{1}{m\tau} \log a_{m\tau}(f, K, N_{1/n}(Q)) \\
&\leq \limsup_{m\to\infty} \frac{1}{m\tau} (2m\tau^n_M \sup f + m \log a_\tau(f, K, N_\varepsilon(Q))) \\
&= \frac{2}{\tau} \tau^n_M \sup f + \frac{1}{\tau} \log a_\tau(f, K, N_\varepsilon(Q)).
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&P_{inv}(f, K, N_{1/n}(Q)) \\
&\leq \lim_{\varepsilon \searrow 0} \liminf_{\tau\to\infty} \quad \frac{2}{\tau} \tau^n_M \sup f + \limsup_{\tau\to\infty} \frac{1}{\tau} \log a_\tau(f, K, N_\varepsilon(Q)) \\
&= \lim_{\varepsilon \searrow 0} \liminf_{\tau\to\infty} \frac{1}{\tau} \log a_\tau(f, K, N_\varepsilon(Q)).
\end{aligned}$$

Since this inequality holds for every $n \in \mathbb{N}$, the assertion follows. ∎

Remark 17. Note that it does not necessarily follow that the limit

$$\lim_{\tau\to\infty} \log a_\tau(f, K, N_\varepsilon(Q))$$

exists for any $\varepsilon > 0$.

6. Invariance pressure for linear control systems. In this section, we prove a main result of this paper. We consider a class of linear control systems given by (A, B) where A is hyperbolic (that is, A has no eigenvalues on the imaginary axis). The control range $U \subset \mathbb{R}^m$ is a compact neighborhood of the origin, and we suppose that the pair (A, B) is controllable (that is, $\operatorname{rank}\big[B\ AB\ \cdots\ A^{d-1}B\big] = d$). Consequently, the system is locally accessible.

From Hinrichsen and Pritchard [5, Theorems 6.2.22 and 6.2.23] (cf. also Colonius and Kliemann [2, Example 3.2.16]) we get the following result on existence and uniqueness of a control set.

THEOREM 18. *Consider a linear control system of the form Σ_{lin} and assume that the pair (A, B) is controllable and the control range U is a compact neighborhood of the origin.*

(i) *Then there is a unique control set D with nonempty interior, it is convex, and satisfies*

$$0 \in \operatorname{int} D \text{ and } D = \mathcal{O}^-(x) \cap \overline{\mathcal{O}^+(x)} \text{ for every } x \in \operatorname{int} D.$$

(ii) *D is closed if and only if $\mathcal{O}^+(x) \subset D$ for all $x \in D$.*

(iii) *The control set D is bounded if and only if A is hyperbolic.*

The following result generalizes and improves [1, Theorem 27] (where the outer invariance pressure was considered). The proof follows Kawan [7, Theorem 4.3], [8, Theorem 5.1], considerably simplified for the linear situation.

THEOREM 19. *Consider a linear control system of the form Σ_{lin} and assume that the pair (A, B) is controllable, that A is hyperbolic, and the control range U is a compact neighborhood of the origin in $\mathbb{R}^m$. Let D be the unique control set with nonempty interior and let $f \in C(U, \mathbb{R})$.*

Then for every compact set $K \subset D$ the pair (K, D) is admissible and

$$P_{inv}(f, K, D) \le \sum_{\lambda\in\sigma(A)} \max\{0, n_\lambda \operatorname{Re}(\lambda)\} + \inf \frac{1}{T}\int_0^T f(\omega(s))ds,$$

where the infimum is taken over all $T > 0$ and all T-periodic controls $\omega(\cdot)$ with a T-periodic trajectory $x(\cdot)$ in $\operatorname{int} D$ such that $\{\omega(t); t \in [0, T]\}$ is contained in a compact subset of $\operatorname{int} U$.

Proof. We will construct a compact subset $K \subset D$ with nonvoid interior such that

$$P_{inv}(f, K, D) \le \sum_{\lambda\in\sigma(A)} \max\{0, n_\lambda \operatorname{Re}(\lambda)\} + \inf \frac{1}{T}\int_0^T f(\omega_0(s))ds.$$

Then the assertion will follow, since every compact subset of D is contained in a compact subset K of D with nonvoid interior and the invariance pressure is independent of the choice of such a set K by Corollary 15.

For the proof consider a τ_0-periodic control $\omega_0(\cdot)$ with τ_0-periodic trajectory as in the statement of the theorem. We can transform A into real Jordan form R without changing the invariance pressure (cf. Example 12), and

$$x_0 = e^{R\tau_0}x_0 + \int_0^{\tau_0} e^{R(\tau_0 - s)}B\omega_0(s)ds. \tag{6.1}$$

Step 1. Choose a basis B of $\mathbb{R}^d$ adapted to the real Jordan structure of R and let $L_1(R), \ldots, L_r(R)$ be the different Lyapunov spaces of R, that is, the sums of the generalized eigenspaces corresponding to eigenvalues with the same real part ρ_j. Then we have the decomposition

$$\mathbb{R}^d = L_1(R) \oplus \cdots \oplus L_r(R).$$

Let $d_j = \dim L_j(R)$ and denote the restriction of R to $L_j(R)$ by R_j. Now take an inner product on $\mathbb{R}^d$ such that the basis B is orthonormal with respect to this inner product and let $\|\cdot\|$ denote the induced norm.

Step 2. We fix some constants: Let S_0 be a real number which satisfies

$$S_0 > \sum_{j=1}^{r} \max(0, d_j\rho_j)$$

and choose $\xi = \xi(S_0) > 0$ such that

$$0 < d\xi < S_0 - \sum_{j=1}^{r} \max(0, d_j\rho_j).$$

Let $\delta \in (0, \xi)$ be chosen small enough such that $\rho_j < 0$ implies $\rho_j + \delta < 0$ for all j. It follows that there exists a constant $c = c(\delta) \geq 1$ such that for all j and for all $k \in \mathbb{N}$

$$e^{tR_j}\| \leq ce^{(\rho_j+\delta)t} \text{ for all } t \geq 0.$$

For every $t > 0$ we define positive integers

$$M_j(t) = \begin{array}{cl} e^{(\rho_j+\xi)t}\rfloor + 1 & \text{if} \quad \rho_j \geq 0, \\ 1 & \text{if} \quad \rho_j < 0. \end{array}$$

Moreover, we define a function $\beta : (0, \infty) \to (0, \infty)$ by

$$\beta(t) = \max_{1 \leq j \leq r} \left[e^{(\rho_j+\delta)t} \frac{\sqrt{d_j}}{M_j(t)} \right], \ t > 0.$$

If $\rho_j < 0$, then $\rho_j + \delta < 0$ and $M_j(t) \equiv 1$. This implies that $e^{(\rho_j+\delta)t}/M_j(t)$ converges to zero for $t \to \infty$. If $\rho_j \geq 0$, we have $M_j(t) \geq e^{(\rho_j+\xi)t}$ and hence

$$e^{(\rho_j+\delta)t} \frac{\sqrt{d_j}}{M_j(t)} \leq e^{(\rho_j+\delta)t} \frac{\sqrt{d_j}}{e^{(\rho_j+\xi)t}} = e^{(\delta-\xi)t}\sqrt{d_j}. \tag{6.2}$$

Since $\delta \in (0, \xi)$, we have $\delta - \xi < 0$ and hence the terms above converge to zero for $t \to \infty$. Thus, also $\beta(t) \to 0$ for $t \to \infty$. Since we assume controllability of (A, B) there exists $C > 0$ such that for every $\lambda \in \mathbb{R}^d$ there is a control $\omega \in L^\infty(0, \tau, \mathbb{R}^m)$ with

$$\varphi(\tau_0, \lambda, \omega) = e^{R\tau_0}\lambda + \int_0^{\tau_0} e^{R(\tau_0 - s)}B\omega(s)ds = 0 \text{ and } \|\omega\|_\infty \leq C\|\lambda\|. \tag{6.3}$$

The inequality follows by the inverse mapping theorem.

For $b_0 > 0$ let $\mathcal{C}$ be the d-dimensional compact cube $\mathcal{C}$ in $\mathbb{R}^d$ centered at the origin with sides of length $2b_0$ parallel to the vectors of the basis B. Choose b_0 small enough such that, with $x_0 := x(0)$,

$$K := x_0 + \mathcal{C} \subset D$$

and $\overline{B(\omega_0(t), Cb_0)} \subset U$ for almost all $t \in [0, \tau_0]$. This is possible, since $x_0 \in \mathrm{int}D$ and almost all values $\omega_0(t)$ are in a compact subset of the interior of U.

Step 3. Let $\varepsilon > 0$ and $\tau = k\tau_0$ with $k \in \mathbb{N}$. We may take $k \in \mathbb{N}$ large enough such that

$$\frac{d}{\tau} \log 2 < \varepsilon. \tag{6.4}$$

Furthermore, we may choose b_0 small enough such that $Cb_0 < \varepsilon$. Partition $\mathcal{C}$ by dividing each coordinate axis corresponding to a component of the jth Lyapunov space $L_j(R)$ into $M_j(\tau)$ intervals of equal length. The total number of subcuboids in this partition of $\mathcal{C}$ is $\prod_{j=1}^r M_j(\tau)^{d_j}$.

Next we will show that it suffices to take $\prod_{j=1}^r M_j(\tau)^{d_j}$ control functions to steer the system from all states in $x_0 + \mathcal{C}$ back to $x_0 + \mathcal{C}$ in time τ such that the controls are within distance ε to ω_0.

Let λ be the center of a subcuboid. By (6.3) there exists $\omega \in L^\infty(0, \tau, \mathbb{R}^m)$ such that

$$\varphi(\tau, \lambda, \omega) = 0 \text{ and } \|\omega\|_\infty \le C\|\lambda\| \le Cb_0 < \varepsilon.$$

Hence $\omega(t) \in U$ for a.a. $t \in [0, \tau]$ and, using (6.1) and linearity, we find that $x_0 + \lambda$ is steered by $\omega_0 + \omega$ in time $\tau = k\tau_0$ to x_0,

$$\varphi(\tau, x_0 + \lambda, \omega_0 + \omega) = \varphi(\tau, x_0, \omega_0) + \varphi(\tau, \lambda, \omega) = x_0. \tag{6.5}$$

Now consider an arbitrary point $x \in \mathcal{C}$. Then it lies in one of the subcuboids and we denote the corresponding center of this subcuboid by λ with associated control ω. We will show that $\omega_0 + \omega$ also steers $x_0 + x$ back to $x_0 + \mathcal{C}$. Observe that

$$\|x - \lambda\| \le \frac{b_0}{M_j(\tau)} \sqrt{d_j}.$$

This implies that

$$e^{\tau R}x - e^{\tau R}\lambda\| \le \quad e^{(k\tau_0 R_j)}\| \, \|x - \lambda\| \le ce^{(\rho_j+\delta)k\tau_0} \frac{b_0}{M_j(k\tau_0)} \sqrt{d_j} \to 0 \text{ for } k \to \infty$$

and, hence, for k large enough $\|e^{\tau R}x - e^{\tau R}\lambda\| \le b_0$. This implies that the solution

$$\varphi(t, x_0 + x, \omega_0 + \omega) = e^{tR}(x_0 + x) + \int_0^t e^{R(t-s)}B\left[\omega_0(s) + \omega(s)\right] ds, t \ge 0,$$

satisfies for k large enough by (6.5) and linearity,

$$\begin{aligned}
&\|\varphi(\tau, x_0 + x, \omega_0 + \omega) - x_0\| \\
&= \quad e^{\tau R}(x_0 + x) + \int_0^\tau e^{R(\tau-s)}B\left[\omega_0(s) + \omega(s)\right] ds - x_0 \\
&\le \|e^{\tau R}(x_0 + x) - e^{\tau R}(x_0 + \lambda)\| + \quad e^{\tau R}(x_0 + \lambda) + \int_0^\tau e^{R(\tau-s)}B\left[\omega_0(s) + \omega(s)\right] ds - x_0 \\
&\le \|e^{\tau R}x - e^{\tau R}\lambda\| + \|\varphi(\tau, x_0 + \lambda, \omega_0 + \omega) - x_0\| \\
&\le ce^{(\rho_j+\delta)k\tau_0} \frac{b_0}{M_j(k\tau_0)} \sqrt{d_j} \le b_0.
\end{aligned}$$

Hence we have proved that $\prod_{j=1}^{r} M_j(\tau)^{d_j}$ control functions are sufficient to steer the system from all states in $x_0 + \mathcal{C}$ back to $x_0 + \mathcal{C}$ in time τ. By the no-return property of control sets it follows that the trajectories do not leave D within the time interval $[0, \tau]$. By iterated concatenation of these control functions we can construct an $(n\tau, K)$-spanning set $\mathcal{S}$ for each $n \in \mathbb{N}$ with cardinality

$$\left(\prod_{j=1}^{r} M_j(\tau)^{d_j}\right)^n = \left(\prod_{j:\rho_j \geq 0} \left\lfloor e^{(\rho_j+\xi)\tau} \right\rfloor + 1\right)^{d_j}\right)^n.$$

It follows that

$$\begin{aligned}
\log a_{n\tau}(f, K, Q) &\leq \log \sum_{\omega \in \mathcal{S}} e^{(S_{n\tau} f)(\omega)} = \log \sum_{\omega \in \mathcal{S}} e^{\int_0^{n\tau} f(\omega(t))dt} \\
&= \log \sum_{\omega \in \mathcal{S}} e^{\int_0^{n\tau} f(\omega_0(t))dt + \int_0^{n\tau} [f(\omega(t)) - f(\omega_0(t))]dt} \\
&\leq \log \left[\sum_{\omega \in \mathcal{S}} e^{\int_0^{n\tau} f(\omega_0(t))dt} \cdot e^{\int_0^{n\tau} \varepsilon dt}\right].
\end{aligned}$$

This implies

$$\begin{aligned}
\frac{1}{n\tau} \log a_{n\tau}(f, K, Q) &\leq \frac{1}{\tau} \sum_{j:\rho_j \geq 0} d_j \log \left. \left\lfloor e^{(\rho_j+\xi)\tau} \right\rfloor + 1\right) + \frac{1}{n\tau} \int_0^{n\tau} f(\omega_0(t))dt + \varepsilon \\
&\leq \frac{1}{\tau} \sum_{j:\rho_j \geq 0} d_j \log(2e^{(\rho_j+\xi)\tau}) + \frac{1}{\tau_0} \int_0^{\tau_0} f(\omega_0(t))dt + \varepsilon \\
&\leq \frac{d}{\tau} \log 2 + \frac{1}{\tau} \sum_{j:\rho_j \geq 0} d_j(\rho_j + \xi)\tau + \frac{1}{\tau_0} \int_0^{\tau_0} f(\omega_0(t))dt + \varepsilon \\
&\leq \frac{d\xi}{\tau} + \sum_{j:\rho_j \geq 0} d_j \rho_j + \frac{1}{\tau_0} \int_0^{\tau_0} f(\omega_0(t))dt + 2\varepsilon \\
&< S_0 + \frac{1}{\tau_0} \int_0^{\tau_0} f(\omega_0(t))dt + 2\varepsilon.
\end{aligned}$$

Here we have also used (6.4). Since ε can be chosen arbitrarily small and S_0 arbitrarily close to $\sum_{j=1}^{r} \max(0, d_j\rho_j)$, the assertion of the theorem follows. ∎

In order to see the relation to Floquet exponents the following simple result is helpful.

PROPOSITION 20. *Consider a τ_0-periodic solution of*

$$\dot{x}(t) = Ax(t) + Bu(t).$$

Then the Floquet exponents of the linearized system (linearized with respect to x) are given by the real parts of the eigenvalues of A and also the algebraic multiplicities coincide. More generally, the Lyapunov exponents are given by

$$\lim_{t\to\infty} \frac{1}{t} \log \|D_x\varphi(t, x, u)y\| = \lim_{n\to\infty} \frac{1}{nT} \log \left\|e^{AnT}x\right\| = \lambda,$$

depending on y.

Proof. We have to analyze the eigenvalues of the linearization of the map $x \mapsto \varphi(\tau_0, x, u) = e^{A\tau_0}x + \int_0^{\tau_0} e^{A(\tau_0-s)}Bu(s)ds$ given by $D_x\varphi(\tau_0, x, u) = e^{A\tau_0}$. Thus the assertion is a consequence of the spectral mapping theorem. ∎

This proposition shows that

$$\sum_{\lambda\in\sigma(A)} \max\{0, n_\lambda \mathrm{Re}(\lambda)\} = \sum_{j=1}^{r} \max\{0, d_j\rho_j\},$$

where $\rho_1, \dots, \rho_r$ are the different Lyapunov exponents with corresponding multiplicities of a periodic solution corresponding to a periodic control. This is the term occurring in the estimate for the invariance entropy in Kawan [8, Theorem 5.1].

COROLLARY 21. *Consider a linear control system of the form Σ_{lin} and assume that the pair (A, B) is controllable, that A is hyperbolic, and the control range U is a compact neighborhood of the origin. Let D be the unique control set, let $f \in C(U, \mathbb{R})$, and suppose that $\min_{\omega\in U} f(\omega) = f(\omega_0)$ with $\omega_0 \in \mathrm{int}U$ and there exists $x_0 \in \mathrm{int}D$ with $Ax_0 + B\omega_0 = 0$.*

Then for every compact set $K \subset D$ with nonempty interior we have that (K, D) is an admissible pair and

$$P_{inv}(f, K, D) = \sum_{\lambda\in\sigma(A)} \max\{0, n_\lambda Re(\lambda)\} + f(\omega_0).$$

Proof. This follows by Theorem 19, since (ω_0, x_0) is a (trivial) periodic solution in $\mathrm{int}U \times \mathrm{int}D$, and for every T-periodic control $\omega(\cdot)$

$$\frac{1}{T}\int_0^T f(\omega(s))ds \geq f(\omega_0).$$

∎

Example 22. Consider the one-dimensional linear system given by the differential equations

$$\dot{x}(t) = ax(t) + \omega(t),\ \omega \in \mathcal{U},$$

where $a > 0$. We assume that the control range $U = [-1, 1]$. Then the compact interval $Q = \left[-\frac{1}{a}, \frac{1}{a}\right]$ is the closure of the unique control set with nonempty interior $D = \mathcal{O}^-(0) = \left(-\frac{1}{a}, \frac{1}{a}\right)$ of this system.

Let $f \in C(U, \mathbb{R})$ such that $f(u_0) = \inf f$ for some $u_0 \in \mathrm{int}U$. Then $x_0 := -\frac{u_0}{a} \in \mathrm{int}D$ and (x_0, u_0) is an equilibrium pair. By Corollary 21 we have

$$P_{inv}(f, K, Q) = \inf f + a.$$

The next example (cf. Sontag [14]) presents an application of outer invariance pressure to a mechanical control system and shows that, in this case, this amount is related to the exponential growth rate of total impulse of external forces acting on the system.

Example 23. Consider a pendulum to which one can apply a torque as an external force (see Figure 1). We assume that friction is negligible, that all of the mass is concentrated at the end, and that the rod has unit length. From Newtons law for rotating objects, there results, in terms of the variable α that describes the counterclockwise angle with respect to the vertical, the second-order nonlinear differential equation

$$m\ddot{\alpha}(t) + mg\sin(\alpha(t)) = u(t), \tag{6.6}$$

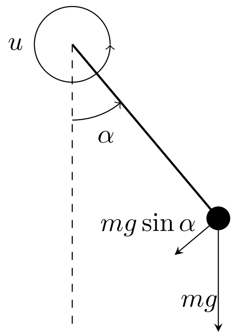

FIG. 1. *Pendulum.*

where m is the mass, g the acceleration due to gravity, and $u(t)$ the value of the external torque at time t (counter-clockwise being positive).

The vertical stationary position $(\alpha, \dot{\alpha}) = (\pi, 0)$ is an equilibrium when the null control $\omega_0 \equiv 0$ is applied, but a small deviation from this will result in an unstable motion. Let us assume that our objective is to apply torques as needed to correct for such deviations. For small $\alpha - \pi$,

$$\sin(\alpha) = -(\alpha - \pi) + r(\alpha - \pi),$$

where $r(t)$ is a function which satisfies $\lim_{t\to 0} \frac{r(t)}{t} = 0$.

Since only small deviations are of interest, we drop the nonlinear part represented by the term $r(t)$. Thus, with $\gamma := \alpha - \pi$ as a new variable, we replace (6.6) by the linear differential equation

$$m\ddot{\gamma}(t) - mg\gamma(t) = \omega(t).$$

If we denote $x_1 = \gamma$ and $x_2 = \dot{\gamma}$, then we obtain

$$\Sigma_1 \ : \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ g & 0 \end{pmatrix}}_{=:A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}}_{=:B} \omega, \ \omega(t) \in U := [-\varepsilon, \varepsilon], \varepsilon > 0.$$

Note that the eigenvalues of A are $\lambda_\pm = \pm\sqrt{g}$. System Σ_1 is via the (time-invariant) conjugacy map (T, id_U) conjugate to (cf. Example 12)

$$\Sigma_2 \ : \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} -\sqrt{g} & 0 \\ 0 & \sqrt{g} \end{pmatrix}}_{=:\widetilde{A}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} \frac{1}{2m} \\ \frac{1}{2m} \end{pmatrix}}_{=:\widetilde{B}} \omega,$$

because $\widetilde{A} = TAT^{-1}$ and $\widetilde{B} = TB$, where

$$T = \frac{1}{2} \begin{pmatrix} -\sqrt{g} & 1 \\ \sqrt{g} & 1 \end{pmatrix} \quad \text{and } T^{-1} = \begin{pmatrix} -\frac{1}{\sqrt{g}} & \frac{1}{\sqrt{g}} \\ 1 & 1 \end{pmatrix}.$$

Note that $\widetilde{A}$ is hyperbolic and the pair $(\widetilde{A}, \widetilde{B})$ is controllable. By Theorem 18, the unique control set $\tilde{D}$ with nonvoid interior of Σ_2 is

$$\tilde{D} = \overline{\mathcal{O}^+(0)} \cap \mathcal{O}^-(0) = \left[-\frac{\varepsilon}{2m\sqrt{g}}, \frac{\varepsilon}{2m\sqrt{g}} \right] \times \left[-\frac{\varepsilon}{2m\sqrt{g}}, \frac{\varepsilon}{2m\sqrt{g}} \right].$$

Then the unique control set with nonvoid interior of Σ_1 is given by $D := T(\tilde{D})$ and one computes

$$D = [-d, d] \times (-d, d) \text{ with } d := \varepsilon \frac{\sqrt{g} + 1}{2m\sqrt{g}}.$$

Here for a compact subset $K \subset Q := D$, a (τ, K, Q)-spanning set $\mathcal{S}$ represents a set of external torques ω that cause the angular position of the pendulum to remain in the interval $[-d, d]$ and such that its angular velocity does not exceed $(-d, d)$ when it starts in K. If $f(u) = |u|$, $u \in U = [-\varepsilon, \varepsilon]$, then $f \in C(U, \mathbb{R})$ and $0 = f(0) = \inf f$. Note that here $(S_\tau f)(\omega)$ represents the impulse of the torque ω until time τ. Hence, the invariance pressure $P_{inv}(f, K, D)$ measures the exponential growth rate of the amount of total impulse required of the external torques acting on the system to remain in D as time tends to infinity. Corollary 21 implies that $P_{inv}(f, K, D) = \sqrt{g} = h_{inv}(K, D)$. The reason is that within the control set D one may steer the system from K arbitrarily close to the equilibrium $0 \in \mathbb{R}^2$ and keep it there with arbitrarily small torque.

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